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Two-dimensional Dendriform Algebra Structures over Any Base Field

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Abstract

In the paper we consider a class of dendriform algebras. A dendriform algebra is a generalization of associative algebra. It is typically defined as an algebra equipped by two multiplication operations satisfying certain compatibility conditions. These algebras have interesting representation theories, leading to insights in both algebra, geometry and physics. We provide a full classification, up to isomorphism, of all two-dimensinal dendriform algebras over any base field. Note that the solution to the classification problem of algebras essentially depends on properties of the base field. The results obtained so far were over the field of complex numbers only. These results come as a particular case of the results in this paper. Moreover, we revise the list of such representatives given earlier over the field of complex numbers.

Keywords: dendriform algebra; classification; Rota–Baxter algebra; two-dimensional algebra.

1 Introduction

Dendriform algebras are an interesting class of algebraic structures that arise in various areas of mathematics, including combinatorics, algebraic topology, and mathematical physics. They can be viewed as a generalization of the concept of associative algebras, and they have unique properties that make them suitable for certain applications. They are closely related to various combinatorial constructs (binary trees), to study tree-like structures and their enumeration. In algebraic topology, dendriform algebras can be employed to study certain types of cohomology theories. Dendriform algebras also appear in the study of operads, which are structures that encode types of operations. Important role dendriform algebras play in Hopf algebras, providing a framework for understanding various algebraic structures in a coherent way. They help in constructing certain Hopf algebras that arise in algebraic combinatorics. In the context of noncommutative geometry, dendriform algebras are used to define certain algebraic invariants and to understand the algebraic structures underlying geometric concepts. Dendriform algebras have also applications in quantum field theory and string theory, where the underlying algebraic structures can help in understanding interactions and symmetries. In Perturbative Quantum Field Theory, dendriform algebras can arise in the context of Feynman diagrams and the organization of perturbative expansions.

Dendriform algebras appeared in [20] due to study of a duality of operads for some classes of algebras. It was proven there that their operad is a Kozul dual to the operad of an algebra equipped by two binary operation. The later was said to be an associative dialgebra. It was found that dendriform algebras have a lot of applications in several branches of mathematics and quantum field theory in connection with so-called Rota–Baxter algebras.

A systematic study of a particular class of linear equations in dendriform algebras is given in [14]. The following results should be mentioned here that have contributed to the development of the theory of dendriform structures. Chapoton [9] (resp. Ronco [26]) discovered that the classical proof of the Cartier–Milnor–Moore theorem [21] (respectively its modern combinatorial proof [23]) could be extended to bialgebras with a dendriform structure, linking dendriform structures with other algebraic structures such as brace and pre-Lie algebras. Aguiar established in [4] unexpected connections with the infinitesimal bialgebra structures studied in [1, 3]. Foissy [17] obtained a striking result in the field of algebraic combinatorics using another dendriform version of the Cartier–Milnor–Moore theorem to prove the Duchamp–Hivert–Thibon conjecture (the Lie algebra of primitive elements of the Malvenuto-Reutenauer Hopf algebra is a free Lie algebra).

A few important results on dendriform algebras have been obtained by Aguiar [2] (the interrelation pre-Poisson and Loday algebras). Ebrahimi-Fard and Guo [13] studied the adjoint functors between the category of Rota–Baxter algebras and the categories of dendriform algebras. Ebrahimi-Fard et al. [15] found a few new combinatorial identities in dendriform algebras. Ebrahimi-Fard et al. [16] generalizes the seminal Cartier–Rota theory of classical Spitzer-type identities for commutative Rota–Baxter algebras. It is shown that associative algebras equipped with a Rota–Baxter operator of arbitrary weight always give such dendriform structures in [12].

There is a classification of low-dimensional complex dendriform algebras in [25]. Dendriform algebra operads, their variations and related problems also are studied. For instance, Bai et al. [5] introduce the concepts of bisuccessor and trisuccessor to study the equivalence of operads of some generalizations of dendriform algebras, Bai et al. [6] generalizes the well-known construction of dendriform algebras and trialgebras from Rota–Baxter algebras to a construction from O-operators, and show that there are bijections between certain equivalence classes of invertible O-operators and equivalence classes of dendriform algebras and trialgebras. The papers [18, 7] are devoted to J-dendriform and L-dendriform, respectively, generalizations of dendriform algebras and their properties. The problems related to deformations of dendriform algebras are studied in [10, 11]. The description of Rota–Baxter operators, Reynolds operators, Nijenhuis operators and averaging operators on $2-$dimensional dendriform algebras over the field of complex numbers is given in [19]. The paper [22] is devoted to study the generalized derivations of complex Left-Symmetric Dialgebras and Rakhimov [24] deals with a complete classification of two-dimensional associative dialgebra structures over any base field.

The paper is organized as follows. Section 1 is an introduction. In Section 2 we move on to cover preliminaries essential to dendriform algebras and facts needed in the later sections. The main results of the paper are in Section 3. Subsection 3.1 contains a complete classification of all dendriform algebra structures on a two-dimensional vector space over fields of characteristic is not two and three. Here, we compare our result with that of obtained earlier over the field of complex numbers and as a result the list of representatives of the isomorphism classes of two-dimensional complex dendriform algebras is revised. Subsections 3.2 and 3.3 contain results on description of dendriform algebra structures on a two-dimensional vector space over fields of characteristic two and three, respectively. Since the idea of the description here is similar to that of the field of characteristic is not two and three the results are given without proofs.

2 Preliminaries

Definition 2.1. *A dendriform algebra (or dendriform dialgebra) over a field $\mathbb{F}$ is a $\mathbb{F}$-vector space $\mathbb{D}$ endowed with two bilinear operations $\prec: \mathbb{D}\times\mathbb{D}\to\mathbb{D}$ and $\succ: \mathbb{D}\times\mathbb{D}\to\mathbb{D}$ for any $\mathrm{a},\mathrm{b},\mathrm{c}\in\mathbb{F}$ subject to the three axioms below:*

$$\begin{aligned}(\mathrm{a}\prec\mathrm{b})\prec\mathrm{c}&=\mathrm{a}\prec(\mathrm{b}\prec\mathrm{c})+\mathrm{a}\prec(\mathrm{b}\succ\mathrm{c}),\\(\mathrm{a}\succ\mathrm{b})\prec\mathrm{c}&=\mathrm{a}\succ(\mathrm{b}\prec\mathrm{c}),\\(\mathrm{a}\prec\mathrm{b})\succ\mathrm{c}+(\mathrm{a}\succ\mathrm{b})\succ\mathrm{c}&=\mathrm{a}\succ(\mathrm{b}\succ\mathrm{c}).\end{aligned}\tag{1}$$

Recall that an associative Rota–Baxter algebra (over a field $\mathbb{F}$) is an associative algebra $(\mathbb{A},\cdot)$ endowed with a linear operator $R:\mathbb{A}\longrightarrow\mathbb{A}$ subject to the following relation,

$$R(\mathrm{a})\cdot R(\mathrm{b})=R\big(R(\mathrm{a})\cdot\mathrm{b}+\mathrm{a}\cdot R(\mathrm{b})+\alpha\mathrm{a}\cdot\mathrm{b}\big),$$

where $\alpha\in\mathbb{F}$. The operator $\tilde{R}:=-\alpha\mathrm{id}-R$ is called a Rota–Baxter operator of weight α. The Rota–Baxter operator $\tilde{R}$ generates the following Dendriform algebra structure on $\mathbb{A}$.

Example 2.1. *Let $(\mathbb{A},R,\alpha)$ be a Rota–Baxter algebra of weight α and define $\prec$ and $\succ$ as follows,*

$$\mathrm{a}\prec\mathrm{b}:=-\mathrm{a}\cdot\tilde{R}(\mathrm{b})\ \ \textit{and}\ \ \mathrm{a}\succ\mathrm{b}:=\tilde{R}(\mathrm{a})\cdot\mathrm{b}.$$

It is immediate to check that $(\mathbb{A},\prec,\succ)$ is a Dendriform algebra.

Example 2.2. *Dendriform algebras of linear operators. Let $\mathbb{A}$ be any algebra of operator-valued functions on the real line, closed under integrals $\int_0^x$. One may wish to consider, for example, smooth $n\times n$ matrix-valued functions. Then, $\mathbb{A}$ is a dendriform algebra for the operations,*

$$(A\prec B)(x):=A(x)\cdot\int_0^x B(t)dt,\qquad(A\succ B)(x):=\int_0^x A(t)dt\cdot B(x),$$

with $A, B \in \mathbb{A}$. Here, let us simply mention that the Rota–Baxter operator on $\mathbb{A}$ giving rise to the dendriform structure is: $R(A)(x) := \int_0^x A(t)dt$. Such a link between (weight zero) Rota–Baxter maps and dendriform algebras first was mentioned in [2].

Example 2.3. *Let $\mathbb{A}$ be an associative algebra and $\mathrm{r} \in \mathbb{A}$ an element such that $\mathrm{r}^2 = 0$. Introduce*

$$\mathrm{a} \prec \mathrm{b} = \mathrm{arb}, \quad \textit{and} \quad \mathrm{a} \succ \mathrm{b} = \mathrm{abr}.$$

This equips a dendriform algebra structure on $\mathbb{A}$.

Example 2.4. *Let $\mathbb{V} = M_2(\mathbb{F})$. Define binary operations as follows,*

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & t_1 \end{pmatrix} \prec \begin{pmatrix} a_2 & b_2 \\ c_2 & t_2 \end{pmatrix} = \begin{pmatrix} -a_1c_2 & a_1a_2 \\ -c_1c_2 & c_1a_2 \end{pmatrix},$$

and

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & t_1 \end{pmatrix} \succ \begin{pmatrix} a_2 & b_2 \\ c_2 & t_2 \end{pmatrix} = \begin{pmatrix} a_1c_2 - c_1a_2 & a_1t_2 - c_1b_2 \\ 0 & 0 \end{pmatrix}.$$

It is straightforward to check that $\mathbb{D} = (\mathbb{V}, \prec, \succ)$ is a dendriform algebra over $\mathbb{F}$.

Let $\mathbb{D} = (V, \prec, \succ)$ and $\mathbb{D}' = (V, \prec', \succ')$ be two dendriform algebras. A morphism between them consists of a linear map $f : \mathbb{D} \longrightarrow \mathbb{D}'$ preserving the structure maps, that is,

$$f(\mathrm{a} \prec \mathrm{b}) = f(\mathrm{a}) \prec' f(\mathrm{b}), \quad \text{and} \quad f(\mathrm{a} \succ \mathrm{b}) = f(\mathrm{a}) \succ' f(\mathrm{b}), \quad \text{for any} \quad \mathrm{a}, \mathrm{b} \in \mathbb{V}.$$

An invertible morphism map $f : \mathbb{D} \longrightarrow \mathbb{D}'$ is said to be isomorphism.

Let $\mathbb{D}$ be $n-$dimensional dendriform algebra over $\mathbb{F}$ and $\mathbf{e} = (e_1, e_2, \dots, e_n)$ be its basis. Then, the bilinear maps $\prec, \succ$ are represented by two $n \times n^2$ matrices (called the matrices of structure constants, shortly MSC),

$$A = \begin{pmatrix} a_{11}^1 & a_{12}^1 & \dots & a_{1n}^1 & a_{21}^1 & a_{22}^1 & \dots & a_{2n}^1 & \dots & a_{n1}^1 & a_{n2}^1 & \dots & a_{nn}^1 \\ a_{11}^2 & a_{12}^2 & \dots & a_{1n}^2 & a_{21}^2 & a_{22}^2 & \dots & a_{2n}^2 & \dots & a_{n1}^2 & a_{n2}^2 & \dots & a_{nn}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{11}^n & a_{12}^n & \dots & a_{1n}^n & a_{21}^n & a_{22}^n & \dots & a_{2n}^n & \dots & a_{n1}^n & a_{n2}^n & \dots & a_{nn}^n \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11}^1 & b_{12}^1 & \dots & b_{1n}^1 & b_{21}^1 & b_{22}^1 & \dots & b_{2n}^1 & \dots & b_{n1}^1 & b_{n2}^1 & \dots & b_{nn}^1 \\ b_{11}^2 & b_{12}^2 & \dots & b_{1n}^2 & b_{21}^2 & b_{22}^2 & \dots & b_{2n}^2 & \dots & b_{n1}^2 & b_{n2}^2 & \dots & b_{nn}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{11}^n & b_{12}^n & \dots & b_{1n}^n & b_{21}^n & b_{22}^n & \dots & b_{2n}^n & \dots & b_{n1}^n & b_{n2}^n & \dots & b_{nn}^n \end{pmatrix},$$

as follows,

$$\mathrm{e}_i \prec \mathrm{e}_j = \sum_{k=1}^{n} a_{ij}^k \mathrm{e}_k, \quad \text{and} \quad \mathrm{e}_i \succ \mathrm{e}_j = \sum_{k=1}^{n} b_{ij}^k \mathrm{e}_k, \quad \text{where} \quad i, j = 1, 2, \dots, n.$$

Therefore, the products $\prec, \succ$ on $\mathbb{D}$ with respect to the basis $\mathbf{e}$ are written as follows,

$$\mathrm{a} \prec \mathrm{b} = \mathbf{e}A(a \otimes b), \quad \text{and} \quad \mathrm{a} \succ \mathrm{b} = \mathbf{e}B(a \otimes b), \tag{2}$$

for any $\mathrm{a} = \mathbf{e}a$, $\mathrm{b} = \mathbf{e}b$, where $a = (a_1, a_2, \ldots, a_n)^T$, and $b = (b_1, b_2, \ldots, b_n)^T$ are column coordinate vectors of a and b, respectively, and

$$a \otimes b = (a_1b_1, a_1b_2, \ldots, a_1b_n, a_2b_1, a_2b_2, \ldots, a_2b_n, \ldots, a_nb_1, a_nb_2, \ldots, a_nb_n)^T,$$

stands for the tensor (Kronecker) product of the coordinate vectors a and b.

Then,

$$\begin{aligned}
&(\mathrm{a} \prec \mathrm{b}) \prec \mathrm{c} = \mathbf{e}A\big(A(a \otimes b) \otimes c\big), && \mathrm{a} \prec (\mathrm{b} \prec \mathrm{c}) = \mathbf{e}A\big(a \otimes A(b \otimes c)\big),\\
&\mathrm{a} \prec (\mathrm{b} \succ \mathrm{c}) = \mathbf{e}A\big(a \otimes B(b \otimes c)\big), && (\mathrm{a} \succ \mathrm{b}) \prec \mathrm{c} = \mathbf{e}A\big(B(a \otimes b) \otimes c\big),\\
&\mathrm{a} \succ (\mathrm{b} \prec \mathrm{c}) = \mathbf{e}B\big(a \otimes (A(b \otimes c))\big), && (\mathrm{a} \prec \mathrm{b}) \succ \mathrm{c} = \mathbf{e}B\big(A(a \otimes b) \otimes c\big),\\
&(\mathrm{a} \succ \mathrm{b}) \succ \mathrm{c} = \mathbf{e}B\big(B(a \otimes b) \otimes c\big), && \mathrm{a} \succ (\mathrm{b} \succ \mathrm{c}) = \mathbf{e}B\big(a \otimes (B(b \otimes c))\big),
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
\mathrm{a} \prec (\mathrm{b} \prec \mathrm{c}) + \mathrm{a} \prec (\mathrm{b} \succ \mathrm{c}) &= \mathbf{e}\Big(A(a \otimes A(b \otimes c)) + A(a \otimes B(b \otimes c))\Big),\\
(\mathrm{a} \prec \mathrm{b}) \succ \mathrm{c} + (\mathrm{a} \succ \mathrm{b}) \succ \mathrm{c} &= \mathbf{e}\Big(B(A(a \otimes b) \otimes c) + B(B(a \otimes b) \otimes c)\Big).
\end{aligned} \tag{4}$$

Therefore, the dendriform algebra axioms (1) in terms of the structure constants can be rewritten as below,

$$\begin{aligned}
A(A(a \otimes b) \otimes c) &= A(a \otimes A(b \otimes c)) + A(a \otimes B(b \otimes c)),\\
A(B(a \otimes b) \otimes c) &= B(a \otimes (A(b \otimes c)),\\
B(A(a \otimes b) \otimes c) + B(B(a \otimes b) \otimes c)) &= B(a \otimes (B(b \otimes c)).
\end{aligned} \tag{5}$$

The equalities (5) in terms of matrices involved are rewritten as follows,

$$\begin{aligned}
A(A \otimes I) &= A(I \otimes A) + A(I \otimes B),\\
A(B \otimes I) &= B(I \otimes A),\\
B(A \otimes I) + B(B \otimes I) &= B(I \otimes B),
\end{aligned} \tag{6}$$

where I is the $n \times n$ identity matrix.

Further we assume that the basis $\mathbf{e}$ is fixed and we do not make a difference between the dendriform algebra $\mathbb{D}$ and its MSC as an ordered pair of matrices $\{A, B\}$.

Let now consider all these in two-dimensional case, i.e.,

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix},$$

and write the equalities (6) in terms of the elements of MSC A and B as follows, AXIOM 1:

$$
\begin{aligned}
\alpha_1\gamma_1+\alpha_2(\beta_1+\delta_1)-\alpha_3\beta_1&=0,\\
(\beta_2+\delta_2)\alpha_2+\alpha_1\gamma_2-\alpha_4\beta_1&=0,\\
\alpha_1(\alpha_2-\alpha_3-\gamma_3)-\alpha_2(\beta_3+\delta_3)+\alpha_3\beta_2&=0,\\
\alpha_2^2-(\beta_4+\delta_4)\alpha_2-\alpha_1(\alpha_4+\gamma_4)+\alpha_4\beta_2&=0,\\
(\beta_3-\gamma_1)\alpha_3-\alpha_4(\beta_1+\delta_1)&=0,\\
\alpha_4(\beta_3-\beta_2-\delta_2)-\alpha_3\gamma_2&=0,\\
\alpha_3^2-(\beta_4-\gamma_3)\alpha_3-\alpha_4(\alpha_1-\beta_3-\delta_3)&=0,\\
(\alpha_2-\alpha_3-\delta_4)\alpha_4-\alpha_3\gamma_4&=0,\\
(\beta_2-\beta_3+\gamma_1)\beta_1+\beta_2\delta_1&=0,\\
(\alpha_2-\beta_4+\gamma_2)\beta_1-\beta_2(\alpha_1-\beta_2-\delta_2)&=0,\\
\beta_1(\alpha_2-\alpha_3-\gamma_3)-\beta_2\delta_3&=0,\\
\beta_1(\alpha_4+\gamma_4)-\beta_2(\alpha_2-\delta_4)&=0,\\
\beta_3^2-(\alpha_1+\gamma_1)\beta_3+(\alpha_3-\beta_4)\beta_1-\beta_4\delta_1&=0,\\
(\alpha_2-\beta_4+\gamma_2)\beta_3+(\beta_2+\delta_2)\beta_4-\alpha_3\beta_2&=0,\\
(\alpha_3+\gamma_3)\beta_3-\alpha_4\beta_1+\beta_4\delta_3&=0,\\
(\beta_2-\beta_3)\alpha_4-\beta_3\gamma_4-\beta_4\delta_4&=0.
\end{aligned}
\tag{7}
$$

AXIOM 2:

$$
\begin{aligned}
\alpha_3\delta_1-\beta_1\gamma_2&=0,\\
\alpha_4\delta_1-\beta_2\gamma_2&=0,\\
(\delta_2-\gamma_1)\alpha_3+\gamma_2(\alpha_1-\beta_3)&=0,\\
(\delta_2-\gamma_1)\alpha_4+\gamma_2(\alpha_2-\beta_4)&=0,\\
\alpha_3\delta_3-\beta_1\gamma_4&=0,\\
\alpha_4\delta_3-\beta_2\gamma_4&=0,\\
(\delta_4-\gamma_3)\alpha_3+\gamma_4(\alpha_1-\beta_3)&=0,\\
(\delta_4-\gamma_3)\alpha_4+\gamma_4(\alpha_2-\beta_4)&=0,\\
(\gamma_1-\delta_2)\beta_1-\delta_1(\alpha_1-\beta_3)&=0,\\
(\gamma_1-\delta_2)\beta_2-\delta_1(\alpha_2-\beta_4)&=0,\\
\beta_1\gamma_2-\alpha_3\delta_1&=0,\\
\beta_2\gamma_2-\alpha_4\delta_1&=0,\\
(\gamma_3-\delta_4)\beta_1-\delta_3(\alpha_1-\beta_3)&=0,\\
(\gamma_3-\delta_4)\beta_2-\delta_3(\alpha_2-\beta_4)&=0,\\
\beta_1\gamma_4-\alpha_3\delta_3&=0,\\
\beta_2\gamma_4-\alpha_4\delta_3&=0.
\end{aligned}
\tag{8}
$$

AXIOM 3:

$$\begin{aligned}
\alpha_1\gamma_1 - \gamma_2\delta_1 + \gamma_3(\beta_1 + \delta_1) = 0,\\
\gamma_4(\beta_1 + \delta_1) - \gamma_1\gamma_2 + \gamma_2(\alpha_1 + \gamma_1 - \delta_2) = 0,\\
\gamma_1(\alpha_2 + \gamma_2 - \gamma_3) - \gamma_2\delta_3 + \gamma_3(\beta_2 + \delta_2) = 0,\\
\gamma_4(\beta_2 + \delta_2) - \gamma_1\gamma_4 + \gamma_2(\alpha_2 + \gamma_2 - \delta_4) = 0,\\
\gamma_1(\alpha_3 + \gamma_3) + \gamma_3(\beta_3 + \delta_3 - \gamma_1) - \gamma_4\delta_1 = 0,\\
\gamma_2(\alpha_3 + \gamma_3) - \gamma_3\gamma_2 + \gamma_4(\beta_3 + \delta_3 - \delta_2) = 0,\\
\gamma_1(\alpha_4 + \gamma_4) + \gamma_3(\beta_4 + \delta_4 - \gamma_3) - \gamma_4\delta_3 = 0,\\
\gamma_2(\alpha_4 + \gamma_4) - \gamma_3\gamma_4 + \gamma_4\beta_4 = 0,\\
\delta_1\alpha_1 - \delta_2\delta_1 + \delta_3(\beta_1 + \delta_1) = 0,\\
\delta_4(\beta_1 + \delta_1) - \gamma_2\delta_1 + \delta_2(\alpha_1 + \gamma_1 - \delta_2) = 0,\\
\delta_1(\alpha_2 + \gamma_2 - \gamma_3) - \delta_2\delta_3 + \delta_3(\beta_2 + \delta_2) = 0,\\
\delta_4(\beta_2 + \delta_2) - \gamma_4\delta_1 + \delta_2(\alpha_2 + \gamma_2 - \delta_4) = 0,\\
\delta_1(\alpha_3 + \gamma_3) + \delta_3(\beta_3 + \delta_3 - \gamma_1) - \delta_4\delta_1 = 0,\\
\delta_2(\alpha_3 + \gamma_3) - \gamma_2\delta_3 + \delta_4(\beta_3 + \delta_3 - \delta_2) = 0,\\
\delta_1(\alpha_4 + \gamma_4) + \delta_3(\beta_4 + \delta_4 - \gamma_3) - \delta_4\delta_3 = 0,\\
\delta_2(\alpha_4 + \gamma_4) - \gamma_4\delta_3 + \beta_4\delta_4 = 0.
\end{aligned} \tag{9}$$

We make use a result given in [8] on classification of two-dimensional algebras with one binary operation over an arbitrary base field. The result was given as follows,

Theorem 2.1. *Any non-trivial 2-dimensional algebra over a field $\mathbb{F}$ is isomorphic to only one of the following listed, by their matrices of structure constants, such algebras,*

1. $Char(\mathbb{F}) \neq 2, 3:$

- $A_1(\mathrm{c}) = \begin{pmatrix} a_1 & a_2 & 1+a_2 & a_4 \\ b_1 & -a_1 & 1-a_1 & -a_2 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_2, a_4, b_1) \in \mathbb{F}^4$.
- $A_2(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & b_2 & 1-a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$ *and* $a_4 \neq 0$.
- $A_3(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & b_2 & 1-a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2a_4 \\ 0 & b_2 & 1-a_1 & 0 \end{pmatrix}$,
  where $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ *and* $a \neq 0$.
- $A_4(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & b_2 & 1 & -1 \end{pmatrix}$, *where* $\mathrm{c} = (b_1, b_2) \in \mathbb{F}^2$.
- $A_5(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 1 & 2a_1-1 & 1-a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = a_1 \in \mathbb{F}$.
- $A_6(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & 1-a_1 & -a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$ *and* $a_4 \neq 0$.
- $A_7(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & 1-a_1 & -a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2a_4 \\ 0 & 1-a_1 & -a_1 & 0 \end{pmatrix}$,
  where $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$.

- $A_8(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & 1 & 0 & -1 \end{pmatrix}$, *where* $\mathrm{c} = b_1 \in \mathbb{F}$.
- $A_9 = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$.
- $A_{10}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1'(a) & 0 & 0 & -1 \end{pmatrix}$,
  where $\mathrm{c} = b_1 \in \mathbb{F}$, *the polynomial* $(b_1t^3 - 3t - 1)(b_1t^2 + b_1t + 1)(b_1^2t^3 + 6b_1t^2 + 3b_1t + b_1 - 2)$ *has no root in* $\mathbb{F}$, $a \in \mathbb{F}$ *and* $b_1'(t) = \dfrac{(b_1^2t^3 + 6b_1t^2 + 3b_1t + b_1 - 2)^2}{(b_1t^2 + b_1t + 1)^3}$.
- $A_{11}(\mathrm{c}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ b_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3b_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$, *where the polynomial* $b_1 - t^3$ *has no root in* $\mathbb{F}$, $a, \mathrm{c} = b_1 \in \mathbb{F}$ *and* $a, b_1 \neq 0$.
- $A_{12}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ a^2b_1 & 0 & 0 & -1 \end{pmatrix}$, *where* $a, \mathrm{c} = \beta_1 \in \mathbb{F}$ *and* $a \neq 0$.
- $A_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

2. $Char(\mathbb{F}) = 2:$

- $A_{1,2}(\mathrm{c}) = \begin{pmatrix} a_1 & a_2 & 1+a_2 & a_4 \\ b_1 & a_1 & 1+a_1 & a_2 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_2, a_4, b_1) \in \mathbb{F}^4$.
- $A_{2,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & b_2 & 1+a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$ *and* $a_4 \neq 0$.
- $A_{2,2}(a_1, 0, 1) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 1 & 1 & 1+a_1 & 0 \end{pmatrix}$, *where* $a_1 \in \mathbb{F}$.
- $A_{3,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & b_2 & 1+a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2a_4 \\ 0 & b_2 & 1+a_1 & 0 \end{pmatrix}$,
  where $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ *and* $a \neq 0$.
- $A_{4,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 1 & 1 & 0 \\ b_1 & b_2 & 1+a_1 & 1 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 1 & 1 & 0 \\ b_1 + (1+b_2)a + a^2 & b_2 & 1+a_1 & 1 \end{pmatrix}$,
  where $\mathrm{c} = (a_1, b_1, b_2) \in \mathbb{F}^3$.
- $A_{5,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & 1+a_1 & a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$ *and* $a_4 \neq 0$.
- $A_{5,2}(1, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$.
- $A_{6,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & 1+a_1 & a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2a_4 \\ 0 & 1+a_1 & a_1 & 0 \end{pmatrix}$,
  where $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$.
- $A_{7,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 1 & 1 & 0 \\ b_1 & 1+a_1 & a_1 & 1 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 1 & 1 & 0 \\ b_1 + aa_1 + a + a^2 & 1+a_1 & a_1 & 1 \end{pmatrix}$,
  where $\mathrm{c} = (a_1, b_1) \in \mathbb{F}^2$ *and* $a \in \mathbb{F}$.

- $A_{8,2}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1^{'}(a) & 0 & 0 & 1 \end{pmatrix}$, *where the polynomial* $(b_1t^3+t+1)(b_1t^2+b_1t+1)$ *has no root in* $\mathbb{F}$, $a \in \mathbb{F}$ *and* $b_1^{'}(t) = \dfrac{(b_1^2t^3+b_1t+b_1)^2}{(b_1t^2+b_1t+1)^3}$.

- $A_{9,2}(\mathrm{c}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ b_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3b_1^{\pm1} & 0 & 0 & 0 \end{pmatrix}$, *where* $a, \mathrm{c} = b_1 \in \mathbb{F}$ *and* $a \neq 0$, *the polynomial* $b_1 + t^3$ *has no root in* $\mathbb{F}$.

- $A_{10,2}(\mathrm{c}) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ b_1 & 1 & 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 & 1 & 0 \\ b_1+a+a^2 & 1 & 1 & 1 \end{pmatrix}$, *where* $a, \mathrm{c} = b_1 \in \mathbb{F}$.

- $A_{11,2}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(b_1+a^2) & 0 & 0 & 1 \end{pmatrix}$, *where* $a, b \in \mathbb{F}$ *and* $b \neq 0$.

- $A_{12,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

3. $Char(\mathbb{F}) = 3:$

- $A_{1,3}(\mathrm{c}) = \begin{pmatrix} a_1 & a_2 & a_2+1 & a_4 \\ b_1 & -a_1 & 1-a_1 & -a_2 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_2, a_4, b_1) \in \mathbb{F}^4$.

- $A_{2,3}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & b_2 & 1-a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$ *and* $a_4 \neq 0$.

- $A_{3,3}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & b_2 & 1-a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2a_4 \\ 0 & b_2 & 1-a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ *and* $a \neq 0$.

- $A_{4,3}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & b_2 & 1 & -1 \end{pmatrix}$, *where* $\mathrm{c} = (b_1, b_2) \in \mathbb{F}^2$.

- $A_{5,3}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 1 & 2a_1-1 & 1-a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = a_1 \in \mathbb{F}$.

- $A_{6,3}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & 1-a_1 & -a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$ *and* $a_4 \neq 0$.

- $A_{7,3}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & 1-a_1 & -a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2a_4 \\ 0 & 1-a_1 & -a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$.

- $A_{8,3}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & 1 & 0 & -1 \end{pmatrix}$, *where* $\mathrm{c} = b_1 \in \mathbb{F}$.

- $A_{9,3}(b_1) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1'(a) & 0 & 0 & -1 \end{pmatrix}$, *where the polynomial* $(b_1-t^3)(b_1t^2+b_1t+1)(b_1^2t^3+b_1-2)$ *has no root in* $\mathbb{F}$, $a \in \mathbb{F}$ *and* $b_1^{'}(t) = \dfrac{(b_1^2t^3+b_1-2)^2}{(b_1t^2+b_1t+1)^3}$.

- $A_{10,3}(\mathrm{c}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ b_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3b_1^{\pm1} & 0 & 0 & 0 \end{pmatrix}$, *where the polynomial* $b_1 - t^3$ *has no root,* $a, \mathrm{c} = b_1 \in \mathbb{F}$ *and* $a, b_1 \neq 0$.

- $A_{11,3}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ a^2 b_1 & 0 & 0 & -1 \end{pmatrix}$, *where* $a, \mathrm{c} = b_1 \in \mathbb{F}, a \neq 0.$
- $A_{12,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix}.$
- $A_{13,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$

We also need the automorphism groups of some of the algebras above to remove the redundance dendriform algebras in the proofs of Theorems 3.1, 3.3 and 3.4.

Lemma 2.1. *The automorphism groups of 2-dimensional algebras corresponding to the left operation in Theorems 3.1, 3.3 and 3.4, generating more than one dendriform algebra, are given as follows,*

1. $Char(\mathbb{F}) \neq 2, 3:$
   - $Aut\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \ t \in \mathbb{F}, \ t \neq 0\right\}.$
   - $Aut\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \ t \in \mathbb{F}, \ t \neq 0\right\}.$
   - $Aut\left(\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix}, \ z, t \in \mathbb{F}, \ t \neq 0\right\}.$
2. $Char(\mathbb{F}) = 2:$
   - $Aut\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix}, \ z, t \in \mathbb{F}, \ t \neq 0\right\}.$
   - $Aut\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \ t \in \mathbb{F}, \ t \neq 0\right\}.$
   - $Aut\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \ t \in \mathbb{F}, \ t \neq 0\right\}.$
3. $Char(\mathbb{F}) = 3:$
   - $Aut\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \ t \in \mathbb{F}, \ t \neq 0\right\}.$
   - $Aut\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \ t \in \mathbb{F}, \ t \neq 0\right\}.$
   - $Aut\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \ t \in \mathbb{F}, \ t \neq 0\right\}.$
   - $Aut\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 0 \\ t-1 & t \end{pmatrix}, \ t \in \mathbb{F}, \ t \neq 0\right\}.$

3 Results

This section and the sections in row are devoted to the classification of two-dimensional dendriform algebras over fields of the characteristic is not two and three, is two and is three. Here, for each algebra of the parts ($Char(\mathbb{F}) \neq 2,3, Char(\mathbb{F}) = 2$ and $Char(\mathbb{F}) = 3$) of Theorem 2.1 we give the systems of equations corresponding to AXIOMS 1, 2 and 3, and provide their solutions, if any, then, we get rid of the algebras occur more than once.

3.1 Classification of two-dimensional dendriform algebras over a field of characteristic is not two and three.

Theorem 3.1. *Any non-trivial 2-dimensional dendriform algebra over a field $\mathbb{F}$, ($Char(\mathbb{F}) \neq 2,3$) is isomorphic to only one of the following listed by their matrices of structure constants, such algebras:*

1. *Dendriform algebra generated by A_{13}:*
   - $\mathbb{D}^1_{13}(\delta_1) := \left\{A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix}, \ \delta_1 \in \mathbb{F}, \delta_1 \neq 0\right\}.$
2. *Dendriform algebra generated by A_7:*
   - $\mathbb{D}^2_7 := \left\{A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right\}.$
3. *Dendriform algebra generated by A_6:*
   - $\mathbb{D}^3_6 := \left\{A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right\}.$
4. *Dendriform algebras generated by A_3:*
   - $\mathbb{D}^4_3 := \left\{A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}.$
   - $\mathbb{D}^5_3(\delta_4) := \left\{A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta_4 \end{pmatrix}, \ \delta_4 \in \{0,1\}\right\}.$
   - $\mathbb{D}^6_3 := \left\{A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right\}.$
   - $\mathbb{D}^7_3 := \left\{A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}.$
   - $\mathbb{D}^8_3 := \left\{A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right\}.$
   - $\mathbb{D}^9_3(\delta_4) := \left\{A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta_4 \end{pmatrix}, \ \delta_4 \in \{0,1\}\right\}.$
   - $\mathbb{D}^{10}_3(\delta_4) := \left\{A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & \delta_4 \end{pmatrix}, \ \delta_4 \in \{0,1\}\right\}.$
   - $\mathbb{D}^{11}_3(\delta_4) := \left\{A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & \delta_4 \end{pmatrix}, \ \delta_4 \in \{0,1\}\right\}.$
   - $\mathbb{D}^{12}_3 := \left\{A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right\}.$
5. *Dendriform algebras generated by A_1:*

- $\mathbb{D}_1^{13} := \left\{ A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \right\}.$
- $\mathbb{D}_1^{14} := \left\{ A = \begin{pmatrix} 2 & -3 & -2 & 4 \\ 1 & -2 & -1 & 3 \end{pmatrix}, \ B = \begin{pmatrix} -1 & 2 & 1 & -2 \\ -1 & 2 & 1 & -2 \end{pmatrix} \right\}.$

Proof. Consider A_{13}, i.e., $D = \{A, B\}$ is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix}.$$

We impose the axioms to get the system of equations and solve the system of equations.

AXIOM 1 implies $\gamma_i = 0$, where $i = 1, 2, 3, 4$. Then, we apply AXIOM 2 and get $\delta_2 = 0$ and $\delta_4 = 0$, i.e.,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & 0 & \delta_3 & 0 \end{pmatrix}.$$

AXIOM 3 gives $\delta_3 = 0$.

Thus, the dendriform algebra generated by A_{13} is given by the following MSC,

$$\mathbb{D}_{13}^1(\delta_1) := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{where} \quad \delta_1 \in \mathbb{F} \right\}.$$

Note that the classes $A_{12} - A_8$ of Theorem 2.1 do not generate a dendriform algebra, in these cases the systems of (7)–(9) are inconsistent.

Consider $A_7(\mathrm{c})$,

$$A_7(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix},$$

where $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$ and $0 \neq a \in \mathbb{F}$.

Let the dialgebra generated by $A = A_7(\mathrm{c})$ be

$$A = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix},$$

where α_1, α_4, γ_1, γ_2, γ_3, γ_4, δ_1, δ_2, δ_3 and δ_4 are unknowns.

The system of (7) is given as follows,

$$\begin{array}{rrr} \alpha_1\gamma_1 = 0, & \alpha_1\gamma_2 = 0, & \alpha_1\gamma_3 = 0, \\ (2\alpha_4 + \gamma_4)\alpha_1 - \alpha_4 = 0, & \alpha_4\delta_1 = 0, & \alpha_4(1 + \delta_2) = 0, \\ \alpha_4(2\alpha_1 - \delta_3) = 0, & \alpha_4\delta_4 = 0, & (1 - \alpha_1)\delta_1 = 0, \\ (1 - \alpha_1)(2\alpha_1 - 1 - \delta_2) = 0, & \delta_3(1 - \alpha_1) = 0, & (1 - \alpha_1)\delta_4 = 0, \\ \alpha_1(2\alpha_1 + \gamma_1) = 0, & \alpha_1\gamma_4 + \alpha_4 = 0. & \end{array} \tag{10}$$

Consider the options: $\alpha_1 = 0$ and $\alpha_1 \neq 0$.

In the former case we have:

AXIOM 1 implies

$$\alpha_1 = 0, \quad \delta_2 = -1, \quad \beta_3 = 0, \quad \alpha_4 = 0.$$

AXIOM 2 implies

$$\begin{array}{lllll} \alpha_1 = 0, & \gamma_1 = -1, & \delta_1 = 0, & \gamma_2 = 0, & \delta_2 = -1, \\ \gamma_3 = 0, & \delta_3 = 0, & \alpha_4 = 0, & \gamma_4 = 0, & \delta_4 = 0. \end{array}$$

AXIOM 3 implies

$$\begin{array}{lllll} \alpha_1 = 0, & \gamma_1 = -1, & \delta_1 = 0, & \gamma_2 = 0, & \delta_2 = -1, \\ \gamma_3 = 0, & \delta_3 = 0, & \alpha_4 = 0, & \gamma_4 = 0, & \delta_4 = 0. \end{array}$$

Thus, in this case the dendriform algebra generated by A_7 is given by the following MSC,

$$\mathbb{D}_7^2 := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \right\}.$$

In the latter case, we have a contradiction.

Consider $A_6(\mathrm{c})$. Then, AXIOM 1 gives the following system of equations for the entries of MSC,

$$\begin{array}{rr} \alpha_1\gamma_1 = 0, & (1-\alpha_1)\delta_1 + 1 + \gamma_1 = 0, \\ \alpha_1\gamma_2 - \alpha_4 = 0, & 2\alpha_1^2 - (\delta_2+3)\alpha_1 + \gamma_2 + \delta_2 + 1 = 0, \\ \alpha_1\gamma_3 = 0, & \delta_3(1-\alpha_1) + \gamma_3 = 0, \\ (2\alpha_4+\gamma_4)\alpha_1 - \alpha_4 = 0, & (1-\alpha_1)\delta_4 + \alpha_4 + \gamma_4 = 0, \\ \alpha_4(1+\delta_1) = 0, & \alpha_1(2\alpha_1+\gamma_1) = 0, \\ \alpha_4(1+\delta_2) = 0, & \alpha_1\gamma_2 = 0, \\ \alpha_4(2\alpha_1 - \delta_3) = 0, & \alpha_1\gamma_3 + \alpha_4 = 0, \\ \alpha_4\delta_4 = 0, & \alpha_1\gamma_4 + \alpha_4 = 0. \end{array} \tag{11}$$

Case 1: $\alpha_1 = 0$. Then, we get

$$\alpha_4 = 0, \quad \gamma_1 = -(1+\delta_1), \quad \gamma_2 = -(1+\delta_2), \quad \gamma_3 = -\delta_3, \quad \text{and} \quad \gamma_4 = -\delta_4.$$

Applying AXIOM 2, we obtain

$$\begin{array}{llllllll} \alpha_1 = 0, & \alpha_2 = 0, & \alpha_3 = 0, & \alpha_4 = 0, & \beta_1 = 1, & \beta_2 = 1, & \beta_3 = 0, & \beta_4 = 0, \\ \gamma_1 = -1, & \gamma_2 = 0, & \gamma_3 = 0, & \gamma_4 = 0, & \delta_1 = 0, & \delta_2 = -1, & \delta_3 = 0, & \delta_4 = 0, \end{array}$$

and AXIOM 3 gives,

$$\begin{array}{llllllll} \alpha_1 = 0, & \alpha_2 = 0, & \alpha_3 = 0, & \alpha_4 = 0, & \beta_1 = 1, & \beta_2 = 1, & \beta_3 = 0, & \beta_4 = 0, \\ \gamma_1 = -1, & \gamma_2 = 0, & \gamma_3 = 0, & \gamma_4 = 0, & \delta_1 = 0, & \delta_2 = -1, & \delta_3 = 0, & \delta_4 = 0. \end{array}$$

Thus, for $\alpha_1 = 0$ the algebra $A_6(\mathrm{c})$ produces a dendriform algebra with MSC,

$$\mathbb{D}_6^3 := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \right\}.$$

Case 2: $\alpha_1 \neq 0$. Then, in (11) above we have

$$\begin{array}{llllll} \alpha_2 = 0, & \alpha_3 = 0, & \alpha_4 = 0, & \beta_1 = 1, & \beta_2 = 1 - \alpha_1, \\ \beta_3 = -\alpha_1, & \beta_4 = 0, & \gamma_1 = 0, & \gamma_2 = 0, & \gamma_3 = 0, & \gamma_4 = 0, \end{array}$$

which yields $\alpha_1 = 0$ that contradicts to $\alpha_1 \neq 0$. Thus, no dendriform algebra appears.

Let $A = A_5(\mathrm{c})$. Applying AXIOM 1, we get

$$\begin{aligned} \alpha_1\gamma_1 &= 0, & 2\alpha_1\delta_3 + \gamma_3 - \delta_3 &= 0, \\ \alpha_1\gamma_2 &= 0, & 2\alpha_1\delta_4 + \gamma_4 - \delta_4 &= 0, \\ \alpha_1\gamma_3 &= 0, & (1-\alpha_1)(1 - 2\alpha_1 - \gamma_1) &= 0, \\ \alpha_1\gamma_4 &= 0, & (1-\alpha_1)\gamma_2 &= 0, \\ (2\delta_1 + 3)\alpha_1 + \gamma_1 - \delta_1 - 2 &= 0, & (1-\alpha_1)\gamma_3 &= 0, \\ 2\alpha_1^2 + (2\delta_2 - 3)\alpha_1 + \gamma_2 - \delta_2 + 1 &= 0, & (1-\alpha_1)\gamma_4 &= 0. \end{aligned} \tag{12}$$

Case 1: $\alpha_1 = 0$. Then, one has

$$\delta_1 = -1, \quad \delta_2 = 1, \quad \delta_3 = 0, \quad \delta_4 = 0, \quad \gamma_1 = 1, \quad \gamma_2 = 0, \quad \gamma_3 = 0, \quad \gamma_4 = 0.$$

Verifying AXIOM 2, we get a contradiction.

Case 2: $\alpha_1 \neq 0$. Then, we obtain

$$\begin{aligned} (2\delta_1 + 3)\alpha_1 - \delta_1 - 2 &= 0, & (2\alpha_1 - 1)(\alpha_1 - 1 + \delta_2) &= 0, \\ 2\alpha_1\delta_3 - \delta_3 &= 0, & 2\alpha_1\delta_4 - \delta_4 &= 0, \\ 2\alpha_1^2 - 3\alpha_1 + 1 &= 0. \end{aligned} \tag{13}$$

From the last equation, we get $\alpha_1 = \frac{1}{2}$ or $\alpha_1 = 1$.

Case 21: $\alpha_1 = \frac{1}{2}$. Then, we get a contradiction with the first equation of the system.

Case 22: Let $\alpha_1 = 1$. In this case, we have $\delta_1 = -1$, $\delta_2 = \delta_3 = \delta_4 = 0$. However, applying AXIOM 2 we get a contradiction again. Therefore, there is no dendriform algebra generated by $A_5(\mathrm{c})$.

Consider $A_4(\mathrm{c})$. Verifying AXIOM 1, we obtain

$$\begin{array}{llll} \delta_1 = 0, & \delta_2 = -\beta_2, & \delta_3 = \beta_2 - 1, & \delta_4 = 2, \\ \gamma_1 = 1, & \gamma_2 = 0, & \gamma_3 = -2, & \gamma_4 = 2, \\ \beta_1 = 0, & \beta_2 = 0, & \delta_2 = 2, \end{array}$$

which produces a contradiction.

Consider $A_3(\mathrm{c})$.

$$A_3(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix},$$

where $c = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ and $a \neq 0$.

Let the dendriform algebra be produced by $A_3(c)$ as follows,

$$A = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix}.$$

AXIOM 1 gives

$$\begin{aligned} \alpha_1\gamma_1 &= 0, & \beta_2\delta_1 &= 0, \\ \alpha_1\gamma_2 &= 0, & \beta_2(\beta_2 - \alpha_1 + \delta_2) &= 0, \\ \alpha_1\gamma_3 &= 0, & \beta_2\delta_3 &= 0, \\ \alpha_1(\alpha_4 + \gamma_4) - \alpha_4\beta_2 &= 0, & \delta_4\beta_2 &= 0, \\ \alpha_4\delta_1 &= 0, & (\alpha_1 - 1)(2\alpha_1 - 1 + \gamma_1) &= 0, \\ \alpha_4(\alpha_1 + \beta_2 + \delta_2 - 1) &= 0, & (\alpha_1 - 1)\gamma_2 &= 0, \\ \alpha_4(1 - 2\alpha_1 + \delta_3) &= 0, & (\alpha_1 - 1)\gamma_3 &= 0, \\ \alpha_4\delta_4 &= 0, & (\beta_2 + \alpha_1 - 1)\alpha_4 + (\alpha_1 - 1)\gamma_4 &= 0. \end{aligned} \tag{14}$$

Case 1: If $\alpha_1 = 0$, then, $\gamma_2 = 0, \gamma_3 = 0, \gamma_1 = 1$, i.e.,

$$A = \begin{pmatrix} 0 & 0 & 0 & \alpha_4 \\ 0 & \beta_2 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix}$$

and

$$\begin{aligned} \alpha_4\beta_2 &= 0, & \alpha_4\delta_1 &= 0, \\ \alpha_4(\beta_2 - 1 + \delta_2) &= 0, & \alpha_4(1 + \delta_3) &= 0, \\ \alpha_4\delta_4 &= 0, & \beta_2\delta_1 &= 0, \\ \beta_2(\beta_2 + \delta_2) &= 0, & \beta_2\delta_3 &= 0, \\ \delta_4\beta_2 &= 0, & (\beta_2 - 1)\alpha_4 - \gamma_4 &= 0. \end{aligned} \tag{15}$$

Case 11: $\alpha_4 = 0$. Then, we obtain $\gamma_4 = 0$ and

$$\beta_2\delta_1 = 0, \quad \beta_2(\beta_2 + \delta_2) = 0, \quad \beta_2\delta_3 = 0, \quad \delta_4\beta_2 = 0.$$

Case 111: $\beta_2 = 0$. Then,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix}.$$

AXIOM 2 yields $\delta_1 = 0$ and $\delta_3 = 0$.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & \delta_4 \end{pmatrix}.$$

Due to AXIOM 3 we obtain $\delta_2(1 - \delta_2) = 0$ and $\delta_4(1 - \delta_2) = 0$.

Case 1111: $\delta_2 = 0$ and $\delta_4 = 0$, i.e., we get a dendriform algebra with MSC as follows,

$$\mathbb{D}_3^4 := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Case 1112: If $\delta_2 = 1$, then, we get

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta_4 \end{pmatrix}.$$

If $\delta_4 \neq 0$ then, the base change $e_1' = e_1$, $e_2' = \delta_4^{-1} e_2$ yields,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Thus, we have the dendriform algebra with MSC as follows:

$$\mathbb{D}_3^5(\delta_4) := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta_4 \end{pmatrix}, \ \delta_4 \in \{0, 1\} \right\}.$$

Case 112: Let $\beta_2 \neq 0$. AXIOM 1 and AXIOM 2 give $\beta_2^2 + \beta_2 = 0$ and this implies $\beta_2 = -1$. Thus, one has a dendriform algebra,

$$\mathbb{D}_3^6 := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}$$

since AXIOM 3 satisfies automatically.

Case 12: If $\alpha_4 \neq 0$, then, there is a contradiction with AXIOM 2.

Case 2: $\alpha_1 \neq 0$. From AXIOM 1, one gets

$$\begin{aligned} \alpha_1\gamma_3 &= 0, & \alpha_1(\alpha_4 + \gamma_4) + \alpha_4\beta_2 &= 0, \\ \alpha_4\delta_1 &= 0, & \alpha_4(\alpha_1 - 1 + \beta_2 + \delta_2) &= 0, \\ \alpha_4(2\alpha_1 - \delta_3 - 1) &= 0, & \alpha_4\delta_4 &= 0, \\ \beta_2\delta_1 &= 0, & \beta_2(\alpha_1 - \beta_2 - \delta_2) &= 0, \\ \beta_2\delta_3 &= 0, & \delta_4\beta_2 &= 0, \\ 2\alpha_1^2 - 3\alpha_1 + 1 &= 0, & (\alpha_1 - 1)\gamma_3 &= 0, \\ (\alpha_1 + \beta_2 - 1)\alpha_4 + \gamma_4(-1 + \alpha_1) &= 0. \end{aligned} \tag{16}$$

From the system above we get two options for α_1 : $\alpha_1 = 1$ or $\alpha_1 = \dfrac{1}{2}$.

Case 21: $\alpha_1 = 1$. This implies

$$\begin{aligned} \gamma_3 &= 0, & (\beta_2 - 1)\alpha_4 - \gamma_4 &= 0, & \alpha_4\delta_1 &= 0, \\ \alpha_4(\beta_2 + \delta_2) &= 0, & \alpha_4(\delta_3 - 1) &= 0, & \alpha_4\delta_4 &= 0, \\ \beta_2\delta_1 &= 0, & \beta_2(\beta_2 - 1 + \delta_2) &= 0, & \beta_2\delta_3 &= 0, \\ \delta_4\beta_2 &= 0, & \alpha_4\beta_2 &= 0. \end{aligned} \tag{17}$$

Case 211: $\alpha_4 = 0$, i.e.,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix}.$$

Case 2111: $\beta_2 = 0$. From AXIOM 2 and AXIOM 3, we obtain $\delta_2(\delta_2 - 1) = 0$, $\delta_4\delta_2 = 0$. This, its turn, gives

$$\{\delta_2 = 0\} \text{ or } \{\delta_2 = 1 \text{ and } \delta_4 = 0\}.$$

Therefore, we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_4 \end{pmatrix}.$$

But if $\delta_4 \neq 0$ then, the base change $e_1' = e_1$, $e_2' = \delta_4^{-1}e_2$ makes $\delta_4 = 1$.

Hence, in this case we obtain the following dendriform algebras,

$$\mathbb{D}_3^7(\delta_4) := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_4 \end{pmatrix}, \ \delta_4 \in \{0, 1\} \right\},$$

(note that if here $\delta_4 = 0$ then, we get a trivial dendriform algebra) and

$$\mathbb{D}_3^8 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\},$$

respectively.

Case 2112: $\beta_2 \neq 0$ case gives a trivial dendriform algebra.

Case 212: if $\alpha_4 \neq 0$, we have a contradiction with AXIOM 2.

Case 22: $\alpha_1 = \dfrac{1}{2}$. Then,

$$\begin{aligned} 2\alpha_4\beta_2 - \alpha_4 - \gamma_4 &= 0, & \alpha_4\delta_1 &= 0, & \alpha_4\delta_3 &= 0, \\ \alpha_4(2\beta_2 - 1 + 2\delta_2) &= 0, & \alpha_4\delta_4 &= 0, & \beta_2\delta_1 &= 0, \\ \beta_2(2\beta_2 - 1 + 2\delta_2) &= 0, & \beta_2\delta_3 &= 0, & \delta_4\beta_2 &= 0. \end{aligned} \tag{18}$$

Let us consider two cases: $\alpha_4 = 0$ and $\alpha_4 \neq 0$.

Case 221: $\alpha_4 = 0$. Then, we have $\gamma_4 = 0$, $\beta_2\delta_1 = 0$, $\beta_2(-1 + 2\beta_2 + 2\delta_2) = 0$, $\beta_2\delta_3 = 0$ and $\delta_4\beta_2 = 0$.

Case 2211: $\beta_2 = 0$. We are left with $\gamma_4 = 0$. It is easy to see that AXIOM 2 satisfies. Let us apply AXIOM 3. Then, one gets

$$\begin{aligned} \delta_1(1 + 2\delta_3 - 2\delta_2) &= 0, \\ \delta_2 - 2\delta_2^2 + 2\delta_4\delta_1 &= 0, \\ \delta_3 + 2\delta_3^2 - 2\delta_4\delta_1 &= 0, \\ \delta_4(1 + 2\delta_3 - 2\delta_2) &= 0. \end{aligned} \tag{19}$$

Case 22111: $\delta_1 = 0$. Then, $\delta_2(1 - 2\delta_2) = 0$, $\delta_3(1 + 2\delta_3) = 0$ and $\delta_4(1 + 2\delta_3 - 2\delta_2) = 0$.

Case 221111: $\delta_2 = 0$. We have $\delta_3(1 + 2\delta_3) = 0$ and $\delta_4(1 + 2\delta_3) = 0$. If $\delta_3 = -\dfrac{1}{2}$, then, we obtain the following MSC,

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \delta_4 \end{pmatrix}$$

But, if $\delta_4 \neq 0$ then, the base change $e'_1 = e_1, e'_2 = \delta_4^{-1}e_2$ from the automorphism group of A yields $\delta_4 = 1$. Therefore, we get

$$\mathbb{D} := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix} \right\}. \tag{20}$$

If $\delta_3 \neq -\frac{1}{2}$, then, $\delta_3 = 0$ and $\delta_4 = 0$. We get a right product trivial dendriform algebra.

Case 221112: $\delta_2 \neq 0$. Then, $\delta_2 = \frac{1}{2}$.

Case 2211121: If $\delta_3 = 0$ then, we get a dendriform algebra,

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \delta_4 \end{pmatrix}, \ \delta_4 \in \mathbb{F}.$$

If $\delta_4 \neq 0$, then, taking $g = \begin{pmatrix} 1 & 0 \\ 0 & \delta_4 \end{pmatrix} \in Aut\left(\left(\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}\right)\right)$ we reduce δ_4 to 1:

$$g^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \delta_4 \end{pmatrix} g^{\otimes 2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{pmatrix}.$$

Thus, we have

$$\mathbb{D}_3^9(\delta_4) := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \delta_4 \end{pmatrix}, \quad \delta_4 \in \{0,1\} \right\}.$$

The base change $e'_1 = 2e_1, e'_2 = e_2$ makes it

$$\mathbb{D}_3^9(\delta_4) := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta_4 \end{pmatrix}, \ \delta_4 \in \{0,1\} \right\}.$$

Case 2211122: If $\delta_3 \neq 0$ then, $\delta_3 = -\frac{1}{2}$ and we obtain

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \delta_4 \end{pmatrix}, \ \delta_4 \in \{0,1\}.$$

If $\delta_4 \neq 0$, then, taking $g = \begin{pmatrix} 1 & 0 \\ 0 & \delta_4 \end{pmatrix} \in Aut\left(\left(\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}\right)\right)$ we reduce δ_4 to 1:

$$g^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \delta_4 \end{pmatrix} g^{\otimes 2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.$$

Therefore, we get

$$\left\{A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \delta_4 \end{pmatrix},\ \delta_4 \in \{0,1\}\right\}.$$

But the base change $e_1' = 2e_1$, $e_2' = e_2$ brings it to

$$\mathbb{D}_3^{10}(\delta_4) := \left\{A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & \delta_4 \end{pmatrix},\ \delta_4 \in \{0,1\}\right\}.$$

Case 22112: $\delta_1 \neq 0$. Then, (19) implies

$$1 + 2\delta_3 - 2\delta_2 = 0, \quad \delta_2 - 2\delta_2^2 + 2\delta_4\delta_1 = 0, \quad \delta_3 + 2\delta_3^2 - 2\delta_4\delta_1 = 0. \tag{21}$$

The solution to (21) is $\left\{\delta_1 \neq 0,\ \delta_2 \text{ is any},\ \delta_3 = \dfrac{2\delta_2 - 1}{2} \text{ and } \delta_4 = \dfrac{\delta_2(2\delta_2-1)}{2\delta_1}\right\}$.

But for the matrix $\begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & \delta_2 - \frac{1}{2} & \dfrac{\delta_2(2\delta_2-1)}{2\delta_1} \end{pmatrix}$ to reduce we have the following options:

(i) If $\delta_2(2\delta_2 - 1) \neq 0$ then, taking

$$g = \begin{pmatrix} 1 & 0 \\ \delta_2 & \dfrac{\delta_2(2\delta_2-1)}{2\delta_1} \end{pmatrix} \in Aut\left(\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}\right)$$

we get

$$g\begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & \delta_2 - \frac{1}{2} & \dfrac{\delta_2(2\delta_2-1)}{2\delta_1} \end{pmatrix}\left(g^{-1}\right)^{\otimes 2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}.$$

(ii) If $\delta_2 = 0$, then, taking

$$g = \begin{pmatrix} 1 & 0 \\ -\dfrac{2\delta_1}{t} & \dfrac{1}{t} \end{pmatrix} \in Aut\left(\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}\right)$$

we get

$$g\begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & 0 & -\frac{1}{2} & 0 \end{pmatrix}\left(g^{-1}\right)^{\otimes 2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

(iii) If $\delta_2 = \frac{1}{2}$, then, taking

$$g = \begin{pmatrix} 1 & 0 \\ \dfrac{2\delta_1}{t} & \dfrac{1}{t} \end{pmatrix} \in Aut\left(\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}\right)$$

we get

$$g\begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & \frac{1}{2} & 0 & 0 \end{pmatrix}\left(g^{-1}\right)^{\otimes 2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

Thus, we get the following dendriform algebras,

$$\mathbb{D}_3^{11}(\delta_4) := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \delta_4 \end{pmatrix}, \ \delta_4 \in \{0, 1\} \right\}.$$

Note that $e_1' = 2e_1, e_2' = e_2$ reduces it to

$$\mathbb{D}_3^{11}(\delta_4) := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & \delta_4 \end{pmatrix}, \ \delta_4 \in \{0, 1\} \right\}.$$

We also have

$$\left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \right\},$$

and it is isomorphic to

$$\mathbb{D}_3^{12} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\},$$

by $e_1' = 2e_1$, $e_2' = e_2$.

Case 2212: $\beta_2 \neq 0$. AXIOM 2 gives $\beta_2 = \frac{1}{2}$. This yields $\delta_2 = 0$ and the dendriform algebra generated is trivial.

Case 222: $\alpha_4 \neq 0$. One has the following solution to AXIOM 1:

$$\begin{array}{llllllll} \alpha_1 = \frac{1}{2}, & \alpha_2 = 0, & \alpha_3 = 0, & \beta_1 = 0, & \beta_2 = 0, & \beta_3 = \frac{1}{2}, & \beta_4 = 0, & \\ \gamma_1 = 0, & \gamma_2 = 0, & \gamma_3 = 0, & \gamma_4 = -\alpha_4, & \delta_1 = 0, & \delta_2 = \frac{1}{2}, & \delta_3 = 0, & \delta_4 = 0. \end{array}$$

But AXIOM 2 produces a contradiction.

Consider A_1. Applying Axiom 1, we get

$$\begin{aligned} \beta_1 - \alpha_1\gamma_1 - \alpha_2\delta_1 &= 0, & (\alpha_2 - \gamma_2)\alpha_1 - \alpha_2\delta_2 + \alpha_4\beta_1 &= 0, \\ (\gamma_3 + 2)\alpha_1 + \alpha_2(\delta_3 + 1) &= 0, & (2\alpha_4 + \gamma_4)\alpha_1 - 2\alpha_2^2 + \alpha_2\delta_4 &= 0, \\ (\alpha_1 + \gamma_1 - 1)\alpha_2 + \alpha_1 + \alpha_4(\beta_1 + \delta_1) + \gamma_1 - 1 &= 0, & \alpha_4(1 - \delta_2) - \gamma_2(1 + \alpha_2) &= 0, \\ 2\alpha_2^2 + (\gamma_3 + 3)\alpha_2 - (2\alpha_1 - \delta_3 - 1)\alpha_4 + \gamma_3 + 1 &= 0, & (\delta_4 + 1)\alpha_4 + \gamma_4(1 + \alpha_2) &= 0, \\ \alpha_1\delta_1 - \beta_1\gamma_1 + \beta_1 &= 0, & 2\alpha_1^2 - \alpha_1\delta_2 + \beta_1(2\alpha_2 + \gamma_2) &= 0, \\ \beta_1(1 + \gamma_3) - \alpha_1\delta_3 &= 0, & (\alpha_2 - \delta_4)\alpha_1 + \beta_1(\alpha_4 + \gamma_4) &= 0, \\ 2\alpha_1^2 + (\gamma_1 - 3)\alpha_1 + (2\beta_1 + \delta_1)\alpha_2 + \beta_1 - \gamma_1 + 1 &= 0, & (\gamma_2 - 1)\alpha_1 + (\delta_2 - 2)\alpha_2 - \gamma_2 &= 0, \\ (\alpha_2 + \gamma_3 + 1)\alpha_1 + (\delta_3 - 1)\alpha_2 + \alpha_4\beta_1 - \gamma_3 - 1 &= 0, & (\alpha_1 - 1)\gamma_4 + \alpha_2\delta_4 - \alpha_4 &= 0. \end{aligned} \tag{22}$$

Case 1: Let $\alpha_2 = 0$. Then,

$$\begin{aligned} \alpha_1\gamma_1 - \beta_1 &= 0, & \alpha_1\gamma_2 - \alpha_4\beta_1 &= 0, \\ \alpha_1\gamma_3 + 2\alpha_1 &= 0, & (2\alpha_4 + \gamma_4)\alpha_1 &= 0, \\ \alpha_1 - 1 + \gamma_1 + \alpha_4(\beta_1 + \delta_1) &= 0, & \alpha_4\delta_2 - \alpha_4 + \gamma_2 &= 0, \\ (2\alpha_1 - \delta_3 - 1)\alpha_4 - \gamma_3 - 1 &= 0, & (\delta_4 + 1)\alpha_4 + \gamma_4 &= 0, \\ \alpha_1\delta_1 - \beta_1\gamma_1 + \beta_1 &= 0, & 2\alpha_1^2 - \alpha_1\delta_2 + \beta_1\gamma_2 &= 0, \\ \beta_1(1 + \gamma_3) - \alpha_1\delta_3 &= 0, & \beta_1(\alpha_4 + \gamma_4) - \alpha_1\delta_4 &= 0, \\ 2\alpha_1^2 + (\gamma_1 - 3)\alpha_1 + \beta_1 - \gamma_1 + 1 &= 0, & (\gamma_2 - 1)\alpha_1 - \gamma_2 &= 0, \\ (\alpha_1 - 1)\gamma_3 + \alpha_4\beta_1 + \alpha_1 - 1 &= 0, & (\alpha_1 - 1)\gamma_4 - \alpha_4 &= 0. \end{aligned} \tag{23}$$

Case 11: $\alpha_1 = 0$. This implies $\beta_1 = 0$, $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = -1$, $\gamma_4 = -\alpha_4$, $\delta_4 = 0$ and

$$\alpha_4\delta_1 = 0, \qquad \alpha_4(1 - \delta_2) = 0, \qquad \text{and} \qquad \alpha_4(\delta_3 + 1) = 0.$$

Case 111: $\alpha_4 = 0$. This implies $\gamma_4 = 0$, $\beta_1 = 0$, $\gamma_1 = 1$, $\gamma_2 = 0$ and $\gamma_3 = -1$, which settles AXIOM 1.

Applying AXIOM 2 we obtain $\delta_1 = 0$, $\delta_2 = 1$, $\delta_3 = 0$, $\delta_4 = -1$ that settles AXIOMS 2 and 3. Thus we have the dendriform algebra,

$$\mathbb{D}_1^{12} := \left\{ A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \right\}.$$

Case 112: If $\alpha_4 \neq 0$, then, $\delta_1 = 0$, $\delta_2 = 1$, $\delta_3 = -1$, $\gamma_4 = -\alpha_4$, $\delta_4 = 0$ and this satisfies AXIOM 1, but AXIOM 2 gives a contradiction.

Case 12: $\alpha_1 \neq 0$. In this case from (23), we obtain $\gamma_3 = -2$, $\gamma_4 = -2\alpha_4$ and

$$\begin{aligned} \alpha_1\gamma_1 - \beta_1 &= 0, & \alpha_1\gamma_2 - \alpha_4\beta_1 &= 0, \\ \alpha_4(\beta_1 + \delta_1) + \alpha_1 + \gamma_1 - 1 &= 0, & \alpha_4\delta_2 - \alpha_4 + \gamma_2 &= 0, \\ 1 + \alpha_4(2\alpha_1 - 1 - \delta_3) &= 0, & \alpha_4\delta_4 - \alpha_4 &= 0, \\ \alpha_1\delta_1 - \beta_1\gamma_1 + \beta_1 &= 0, & 2\alpha_1^2 - \alpha_1\delta_2 + \beta_1\gamma_2 &= 0, \\ \alpha_1\delta_3 + \beta_1 &= 0, & \alpha_1\delta_4 + \alpha_4\beta_1 &= 0, \\ 2\alpha_1^2 + (\gamma_1 - 3)\alpha_1 + \beta_1 - \gamma_1 + 1 &= 0, & (\gamma_2 - 1)\alpha_1 - \gamma_2 &= 0, \\ \alpha_4\beta_1 - \alpha_1 + 1 &= 0, & 2\alpha_1\alpha_4 - \alpha_4 &= 0. \end{aligned} \tag{24}$$

Case 121: $\alpha_4 = 0$. It is easy to see in (24), we get a contradiction.

Case 122: $\alpha_4 \neq 0$. Then, the solution to (7) is

$$\begin{aligned} &\alpha_1 = \frac{1}{2}, && \alpha_2 = 0, && \alpha_3 = 1, && \\ &b_1 = -\frac{1}{2\alpha_4}, && b_2 = -\frac{1}{2}, && b_3 = \frac{1}{2}, && b_4 = 0, \\ &c_1 = -\frac{1}{\alpha_4}, && c_2 = -1, && c_3 = -2, && c_4 = -2\alpha_4, \\ &d_1 = \frac{\alpha_4 + 1}{\alpha_4^2}, && d_2 = \frac{1 + \alpha_4}{\alpha_4}, && d_3 = \frac{1}{\alpha_4}, && d_4 = 1, \end{aligned}$$

but (8) is inconsistent.

Case 2: Consider $\alpha_2 \neq 0$. Then,

$$\begin{aligned}
\alpha_1\gamma_1 + \alpha_2\delta_1 - \beta_1 &= 0, & (\alpha_2 - \gamma_2)\alpha_1 - \delta_2\alpha_2 + \alpha_4\beta_1 &= 0,\\
(\gamma_3 + 2)\alpha_1 + \alpha_2(\delta_3 + 1) &= 0, & (2\alpha_4 + \gamma_4)\alpha_1 - 2\alpha_2^2 + \alpha_2\delta_4 &= 0,\\
(\alpha_1 + \gamma_1 - 1)\alpha_2 + \alpha_1 + \alpha_4(\beta_1 + \delta_1) + \gamma_1 - 1 &= 0, & \alpha_4(1 - \delta_2) - \gamma_2(1 + \alpha_2) &= 0,\\
2\alpha_2^2 + (\gamma_3 + 3)\alpha_2 - (2\alpha_1 - \delta_3 - 1)\alpha_4 + \gamma_3 + 1 &= 0, & (\delta_4 + 1)\alpha_4 + \gamma_4(1 + \alpha_2) &= 0,\\
\alpha_1\delta_1 - \beta_1\gamma_1 + \beta_1 &= 0, & 2\alpha_1^2 - \delta_2\alpha_1 + (2\alpha_2 + \gamma_2)\beta_1 &= 0,\\
\beta_1(1 + \gamma_3) - \alpha_1\delta_3 &= 0, & (\alpha_2 - \delta_4)\alpha_1 + \beta_1(\alpha_4 + \gamma_4) &= 0,\\
2\alpha_1^2 + (\gamma_1 - 3)\alpha_1 + (2\beta_1 + \delta_1)\alpha_2 + \beta_1 - \gamma_1 + 1 &= 0, & (\gamma_2 - 1)\alpha_1 + (\delta_2 - 2)\alpha_2 - \gamma_2 &= 0,\\
(\alpha_2 + \gamma_3 + 1)\alpha_1 + (\delta_3 - 1)\alpha_2 + \alpha_4\beta_1 - \gamma_3 - 1 &= 0, & (\alpha_1 - 1)\gamma_4 + \alpha_2\delta_4 - \alpha_4 &= 0.
\end{aligned} \tag{25}$$

We find δ_i, where $i = 1, 2, 3, 4$:

$$\begin{aligned}
\delta_1 &= -\frac{\alpha_1\gamma_1 - \beta_1}{\alpha_2},\\
\delta_2 &= \frac{\alpha_1\alpha_2 - \alpha_1\gamma_2 + \alpha_4\beta_1}{\alpha_2},\\
\delta_3 &= -\frac{\alpha_1\gamma_3 + 2\alpha_1 + \alpha_2}{\alpha_2},\\
\delta_4 &= -\frac{2\alpha_1\alpha_4 + \alpha_1\gamma_4 - 2\alpha_2^2}{\alpha_2},
\end{aligned} \tag{26}$$

and substitute it into the system to get

$$\begin{aligned}
(1 - \alpha_1 - \gamma_1)\alpha_2^2 + (-\alpha_4\beta_1 - \alpha_1 - \gamma_1 + 1)\alpha_2 + \alpha_4(\alpha_1\gamma_1 - \beta_1) &= 0,\\
\gamma_2\alpha_2^2 - ((1 - \alpha_1)\alpha_4 - \gamma_2)\alpha_2 - \alpha_1\alpha_4\gamma_2 + \alpha_4^2\beta_1 &= 0,\\
2\alpha_2^3 + (\gamma_3 + 3)\alpha_2^2 - (2\alpha_1\alpha_4 - \gamma_3 - 1)\alpha_2 - \alpha_1\alpha_4(\gamma_3 + 2) &= 0,\\
(2\alpha_4 + \gamma_4)\alpha_2^2 + (\alpha_4 + \gamma_4)\alpha_2 - \alpha_4(2\alpha_4 + \gamma_4)\alpha_1 &= 0,\\
(1 - \gamma_1)\beta_1\alpha_2 - \alpha_1^2\gamma_1 + \alpha_1\beta_1 &= 0,\\
(\alpha_2 + \gamma_2)\alpha_1^2 - \alpha_1\alpha_4\beta_1 + \beta_1\alpha_2(2\alpha_2 + \gamma_2) &= 0,\\
(\alpha_1 + \beta_1(1 + \gamma_3))\alpha_2 + \alpha_1^2(\gamma_3 + 2) &= 0,\\
\alpha_1\alpha_2^2 - \beta_1(\alpha_4 + \gamma_4)\alpha_2 - (2\alpha_4 + \gamma_4)\alpha_1^2 &= 0,\\
2\alpha_1^2 - 3\alpha_1 + (2\alpha_2 + 2)\beta_1 - \gamma_1 + 1 &= 0,\\
(\alpha_2 - 1)\alpha_1 + \alpha_4\beta_1 - 2\alpha_2 - \gamma_2 &= 0,\\
(\alpha_2 - 1)\alpha_1 + \alpha_4\beta_1 - 2\alpha_2 - \gamma_3 - 1 &= 0,\\
(2\alpha_1 + 1)\alpha_4 - 2\alpha_2^2 + \gamma_4 &= 0.
\end{aligned} \tag{27}$$

Now we express γ_i, where $i = 1, 2, 3, 4$ as follows,

$$\begin{aligned}
\gamma_1 &= 2\alpha_1^2 - 3\alpha_1 + 2(\alpha_2 + 1)b_1 + 1,\\
\gamma_2 &= (\alpha_2 - 1)\alpha_1 + \alpha_4 b_1 - 2\alpha_2,\\
\gamma_3 &= (\alpha_2 - 1)\alpha_1 + \alpha_4 b_1 - 2\alpha_2 - 1,\\
\gamma_4 &= (2\alpha_1 + 1)\alpha_4 - 2\alpha_2^2,
\end{aligned} \tag{28}$$

substitute into (27) and get

$$\begin{aligned}
(2\alpha_1\alpha_4 - 2\alpha_2^2 - 2\alpha_2 - \alpha_4)(\alpha_1^2 + \alpha_2\beta_1 - \alpha_1 + \beta_1) &= 0,\\
(\alpha_1 - 2)\alpha_2^3 - (\alpha_4\beta_1 - 2)\alpha_2^2 + ((\alpha_1^2 - 3\alpha_1 - \beta_1 + 1)\alpha_4 + \alpha_1)\alpha_2 &\\
+\alpha_4(\beta_1(\alpha_1 - 1)\alpha_4 - \alpha_1^2) &= 0,\\
\alpha_4(\alpha_2 - 1)\alpha_1^2 + (-\alpha_2^3 + \alpha_4^2\beta_1 + \alpha_2 + \alpha_4)\alpha_1 - \alpha_2\alpha_4\beta_1(1 + \alpha_2) &= 0,\\
(2\alpha_2^2 + 2\alpha_2 - 2\alpha_1\alpha_4 + \alpha_4)(\alpha_1\alpha_4 - \alpha_2^2) &= 0,\\
(\alpha_1^2 - \alpha_1 + \beta_1 + \alpha_2\beta_1)(2\alpha_1^2 + 2\alpha_2\beta_1 - \alpha_1) &= 0,\\
(\alpha_2 - 1)\alpha_1^3 - (\alpha_4\beta_1 - \alpha_2)\alpha_1^2 + \beta_1(\alpha_2^2 - \alpha_2 - \alpha_4)\alpha_1 + \alpha_2\alpha_4\beta_1^2 &= 0,\\
(\alpha_2 - 1)\alpha_1^3 - (\alpha_4\beta_1 - 2\alpha_2 + 1)\alpha_1^2 + \alpha_2(\alpha_2\beta_1 - \beta_1 + 1)\alpha_1 &\\
+\alpha_2\alpha_4\beta_1^2 - 2\alpha_2^2\beta_1 &= 0,\\
(2\alpha_1^2 + 2\alpha_2\beta_1 - \alpha_1)(\alpha_1\alpha_4 - \alpha_2^2) &= 0.
\end{aligned} \tag{29}$$

Case 21: $2\alpha_1^2 + 2\alpha_2 b_1 - \alpha_1 = 0$, i.e., $b_1 = \dfrac{\alpha_1 - 2\alpha_1^2}{2\alpha_2}$. Substituting $\beta_1 = \dfrac{\alpha_1 - 2\alpha_1^2}{2\alpha_2}$ into (29), we obtain

$$\begin{aligned}
(2\alpha_1\alpha_4 - 2\alpha_2^2 - 2\alpha_2 - \alpha_4)\alpha_1(2\alpha_1 + \alpha_2 - 1) &= 0,\\
(2\alpha_1\alpha_4 - 2\alpha_2^2 - 2\alpha_2 - \alpha_4)((-\alpha_1 + 2)\alpha_2^2 + \alpha_1\alpha_2 + \alpha_1\alpha_4(\alpha_1 - 1)) &= 0,\\
(2\alpha_1\alpha_4 - 2\alpha_2^2 - 2\alpha_2 - \alpha_4)\alpha_1(\alpha_1\alpha_4 - \alpha_2^2 + \alpha_2) &= 0,\\
(2\alpha_1\alpha_4 - 2\alpha_2^2 - 2\alpha_2 - \alpha_4)(\alpha_1\alpha_4 - \alpha_2^2) &= 0,\\
\alpha_1^2(2\alpha_1\alpha_4 - 2\alpha_2^2 - 2\alpha_2 - \alpha_4) &= 0.
\end{aligned} \tag{30}$$

Here, if $\alpha_1 = 0$, then, we get $\alpha_2 = -1$, $\alpha_3 = 0$, $\alpha_4 = 0$, $\beta_1 = 0$, $\beta_2 = 0$, $\beta_3 = 1$, $\beta_4 = 1$, $\gamma_1 = 1$, $\gamma_2 = 2$, $\gamma_3 = 1$, $\gamma_4 = 2$, $\delta_1 = 0$, $\delta_2 = 0$, $\delta_3 = -1$, $\delta_4 = -2$ and this satisfies AXIOM 1. But AXIOM 2 gives a contradiction.

If $\alpha_1 \neq 0$ simplifying (30), we get

$$2\alpha_2^2 + 2\alpha_2 - 2\alpha_4\alpha_1 + \alpha_4 = 0.$$

Case 211: If $\alpha_1 = \dfrac{1}{2}$, then, $\alpha_2 = -1$. This causes $\beta_1 = 0$, $\beta_2 = -\dfrac{1}{2}$, $\beta_3 = \dfrac{1}{2}$, $\beta_4 = 1$, $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3 = 0$, $\gamma_4 = 2 - 2a_4$, $\delta_1 = 0$, $\delta_2 = 1$, $\delta_3 = 0$ and $\delta_4 = -1$ and this settles AXIOM 1. But AXIOM 2 gives a contradiction.

Case 212: Let $\alpha_1 \neq \dfrac{1}{2}$, then, $\alpha_4 = \dfrac{2\alpha_2(\alpha_2 + 1)}{2\alpha_1 - 1}$. Substitute it into (28) and get

$$\begin{aligned}
\gamma_1 &= -\frac{(\alpha_1 + \alpha_2)(2\alpha_1 - 1)}{\alpha_2},\\
\gamma_2 &= -2(\alpha_1 + \alpha_2),\\
\gamma_3 &= -2(\alpha_1 + \alpha_2) - 1,\\
\gamma_4 &= -\frac{2\alpha_2(2\alpha_1 + 2\alpha_2 + 1)}{2\alpha_1 - 1}.
\end{aligned} \tag{31}$$

Substituting all into (26), we obtain

$$\begin{aligned}\delta_1 &= \frac{\alpha_1(2\alpha_1-1)(2\alpha_1+2\alpha_2-1)}{2\alpha_2^2},\\ \delta_2 &= \frac{(2\alpha_1+2\alpha_2-1)\alpha_1}{\alpha_2},\\ \delta_3 &= \frac{(\alpha_1+\alpha_2)(2\alpha_1-1)}{\alpha_2},\\ \delta_4 &= 2(\alpha_1+\alpha_2).\end{aligned} \tag{32}$$

This satisfies AXIOM 1.

Verifying AXIOM 2, we obtain

$$\begin{aligned} 2\alpha_1+\alpha_2-1=0, &\qquad 2\alpha_1^2-\alpha_2^2-\alpha_1-\alpha_2=0,\\ 4\alpha_1+3\alpha_2+1=0, &\qquad 3\alpha_1+2\alpha_2=0,\end{aligned}$$

i.e., $\alpha_1=2$ and $\alpha_2=-3$. We get $\alpha_1=2$, $\alpha_2=-3$, $\alpha_3=-2$, $\alpha_4=4$, $\beta_1=1$, $\beta_2=-2$, $\beta_3=-1$, $\beta_4=3$, $\gamma_1=-1$, $\gamma_2=2$, $\gamma_3=1$, $\gamma_4=-2$, $\delta_1=-1$, $\delta_2=2$, $\delta_3=1$ and $\delta_4=-2$ as a solution to the systems of (8) and (9). That means,

$$\mathbb{D}_1^{13}:=\left\{A=\begin{pmatrix}2&-3&-2&4\\1&-2&-1&3\end{pmatrix},\quad B=\begin{pmatrix}-1&2&1&-2\\-1&2&1&-2\end{pmatrix}\right\},$$

is the next two-dimensional dendriform algebra.

□

Remark 3.1. *In [25] the representatives of the isomorphism classes of two-dimensional dendriform algebra structures over $\mathbb{C}$ were given as follows:*

Theorem 3.2. *Any two-dimensional dendriform algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$Dend_2^1(\alpha): e_1\prec e_1=e_2,\ e_1\succ e_1=\alpha e_2,\ \alpha\in\mathbb{C}$; ($Dend_2^1(0)$ *is trivial*)

$$\left\{A=\begin{pmatrix}0&0&0&0\\1&0&0&0\end{pmatrix},\quad B=\begin{pmatrix}0&0&0&0\\\alpha&0&0&0\end{pmatrix}\right\}.$$

$Dend_2^2: e_1\prec e_1=e_1,\ e_1\succ e_2=e_2$;

$$\left\{A=\begin{pmatrix}1&0&0&0\\0&0&0&0\end{pmatrix},\quad B=\begin{pmatrix}0&0&0&0\\0&1&0&0\end{pmatrix}\right\}.$$

$Dend_2^3: e_1\prec e_1=e_1,\ e_2\prec e_1=e_2,\ e_1\succ e_2=e_2,\ e_2\succ e_1=-e_2$;

$$\left\{A=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix},\quad B=\begin{pmatrix}0&0&0&0\\0&1&-1&0\end{pmatrix}\right\}.$$

$Dend_2^4: e_2\prec e_1=e_2,\ e_1\succ e_1=e_1$;

$$\left\{A=\begin{pmatrix}0&0&0&0\\0&1&0&0\end{pmatrix},\quad B=\begin{pmatrix}1&0&0&0\\0&0&0&0\end{pmatrix}\right\}.$$

$Dend_2^5 : e_1 \prec e_2 = -e_2, e_2 \prec e_1 = e_2, e_1 \succ e_1 = e_1, e_1 \succ e_2 = e_2;$

$$\left\{A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right\}.$$

$Dend_2^6 : e_1 \prec e_1 = e_1, e_2 \succ e_2 = e_2;$

$$\left\{A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}.$$

$Dend_2^7 : e_1 \prec e_2 = -e_2, e_2 \prec e_2 = e_2, e_1 \succ e_1 = e_1, e_1 \succ e_2 = e_2;$

$$\left\{A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right\}.$$

$Dend_2^8 : e_1 \prec e_1 = e_1 + e_2, e_1 \prec e_2 = -e_2, e_2 \prec e_2 = e_2, e_1 \succ e_1 = -e_2, e_1 \succ e_2 = e_2;$

$$\left\{A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}\right\}.$$

$Dend_2^9 : e_1 \prec e_1 = e_1, e_2 \prec e_1 = e_2, e_2 \succ e_1 = -e_2, e_2 \succ e_2 = e_2;$

$$\left\{A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}\right\}.$$

$Dend_2^{10} : e_1 \prec e_1 = -e_2, e_2 \prec e_1 = e_2, e_1 \succ e_1 = e_1 + e_2, e_2 \succ e_1 = -e_2, e_2 \succ e_2 = e_2;$

$$\left\{A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix}\right\}.$$

$Dend_2^{11} : e_2 \prec e_1 = e_2, e_1 \succ e_1 = e_1, e_1 \succ e_2 = e_2;$

$$\left\{A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right\}.$$

$Dend_2^{12} : e_1 \prec e_1 = e_1, e_2 \prec e_1 = e_2, e_1 \succ e_2 = e_2$

$$\left\{A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right\}.$$

Here is a comparison the list of Theorem 3.1 with that of [25] in Table 1.

Table 1: C omparison the list of Theorem 3.1 with that of [25].

Algebra from Theorem 3.1	Algebra from [25]	Omissions in [25]
$\mathbb{D}^1_{13}(\delta_1),\ \delta_1 \in \mathbb{F}$	$Dend^1_2(\alpha) = \mathbb{D}^1_{13}(\delta_1),\ \ \delta_1 = \alpha$	
$\mathbb{D}^2_7$		$\mathbb{D}^2_7$
$\mathbb{D}^3_6$		$\mathbb{D}^3_6$
$\mathbb{D}^4_3$		$\mathbb{D}^4_3$
$\mathbb{D}^5_3(\delta_4),\ \delta_4 \in \{0,1\}$	$Dend^{10}_2 = \mathbb{D}^5_3(1)\ Dend^{11}_2 = \mathbb{D}^5_3(0)$	
$\mathbb{D}^6_3$	$Dend^5_2 = \mathbb{D}^6_3$	
$\mathbb{D}^7_3(\delta_4)$	$Dend^6_2 = \mathbb{D}^7_3(1)$	$\mathbb{D}^7_3(0)$
$\mathbb{D}^8_3$	$Dend^2_2 = \mathbb{D}^8_3$	
$\mathbb{D}^9_3(\delta_4),\ \delta_4 \in \{0,1\}$	$Dend^{12}_2 = \mathbb{D}^9_3(0)$	$\mathbb{D}^9_3(1)$
$\mathbb{D}^{10}_3(\delta_4),\ \delta_4 \in \{0,1\}$	$Dend^3_2 = \mathbb{D}^{10}_3(0)$	$\mathbb{D}^{10}_3(1)$
$\mathbb{D}^{11}_3(\delta_4),\ \delta_4 \in \{0,1\}$	$Dend^9_2 = \mathbb{D}^{11}_3(1)$	$\mathbb{D}^{11}_3(0)$
$\mathbb{D}^{12}_3$		$\mathbb{D}^{12}_3$
$\mathbb{D}^{13}_1$	$Dend^7_2 \cong \mathbb{D}^{13}_1 \text{ via } \begin{cases} e'_1 = -e_2, \\ e'_2 = e_1 + e_2. \end{cases}$	
$\mathbb{D}^{14}_1$	$Dend^8_2 \cong \mathbb{D}^{14}_1 \text{ via } \begin{cases} e'_1 = e_1, \\ e'_2 = e_1 + e_2. \end{cases}$	

It was found that the algebra $Dend^4_2$ given in [25] is not dendriform algebra at all.

The next two sections are devoted to the classification problem of dendriform algebra structures on two-dimensional vector spaces over fields of the characteristics two and three. The results are given without proof since they can be obtained by small modifications of the proof for the field of characteristic neither two nor three and using the parts of Theorem 2.1 and Lemma 2.1 concerning the fields of the characteristics two and three, respectively.

3.2 Classification of two-dimensional dendriform algebras over a field of characteristic is two

Theorem 3.3. *Any non-trivial 2-dimensional dendriform algebra over a field $\mathbb{F}$, $(Char(\mathbb{F}) = 2)$ is isomorphic to only one of the following listed by their matrices of structure constants, such algebras:*

1. *Dendriform algebra generated by $A_{12,2}$:*

   - $\mathbb{D}^1_{12,2}(\delta_1) = \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix},\ \delta_1 \in \mathbb{F},\ \delta_1 \neq 0 \right\}.$

2. *Dendriform algebra generated by $A_{9,2}$:*

- $\mathbb{D}^2_{9,2}(\gamma_4)=\left\{A=\begin{pmatrix}0&0&0&1\\0&0&0&0\end{pmatrix},\ B=\begin{pmatrix}0&0&0&\gamma_4\\0&0&0&0\end{pmatrix},\ \gamma_4\in\mathbb{F},\gamma_4\neq 0\right\}.$

3. *Dendriform algebras generated by $A_{6,2}$:*

- $\mathbb{D}^3_{6,2}=\left\{A=\begin{pmatrix}0&0&0&0\\0&1&0&0\end{pmatrix},\ B=\begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix}\right\}.$
- $\mathbb{D}^4_{6,2}=\left\{A=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix},\ B=\begin{pmatrix}0&0&0&0\\0&0&1&0\end{pmatrix}\right\}.$
- $\mathbb{D}^5_{6,2}=\left\{A=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix},\ B=\begin{pmatrix}0&0&0&0\\0&1&0&0\end{pmatrix}\right\}.$
- $\mathbb{D}^6_{6,2}=\left\{A=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix},\ B=\begin{pmatrix}0&0&0&0\\0&1&1&0\end{pmatrix}\right\}.$
- $\mathbb{D}^7_{6,2}=\left\{A=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix},\ B=\begin{pmatrix}0&0&0&0\\0&0&1&0\end{pmatrix}\right\}.$
- $\mathbb{D}^8_{6,2}=\left\{A=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix},\ B=\begin{pmatrix}0&0&0&0\\0&1&0&0\end{pmatrix}\right\}.$
- $\mathbb{D}^9_{6,2}=\left\{A=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix},\ B=\begin{pmatrix}0&0&0&0\\0&0&1&1\end{pmatrix}\right\}.$

4. *Dendriform algebras generated by $A_{3,2}$:*

- $\mathbb{D}^{10}_{3,2}(\delta_4)=\left\{A=\begin{pmatrix}0&0&0&0\\0&0&1&0\end{pmatrix},\ B=\begin{pmatrix}1&0&0&0\\0&1&0&\delta_4\end{pmatrix},\ \delta_4\in\{0,1\}\right\}.$
- $\mathbb{D}^{11}_{3,2}=\left\{A=\begin{pmatrix}0&0&0&0\\0&0&1&0\end{pmatrix},\ B=\begin{pmatrix}1&0&0&0\\0&0&0&0\end{pmatrix}\right\}.$
- $\mathbb{D}^{12}_{3,2}=\left\{A=\begin{pmatrix}0&0&0&0\\0&1&1&0\end{pmatrix},\ B=\begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix}\right\}.$
- $\mathbb{D}^{13}_{3,2}=\left\{A=\begin{pmatrix}1&0&0&0\\0&0&0&0\end{pmatrix},\ B=\begin{pmatrix}0&0&0&0\\0&0&0&1\end{pmatrix}\right\}.$
- $\mathbb{D}^{14}_{3,2}=\left\{A=\begin{pmatrix}1&0&0&0\\0&0&0&0\end{pmatrix},\ B=\begin{pmatrix}0&0&0&0\\0&1&0&0\end{pmatrix}\right\}.$

5. *Dendriform algebra generated by $A_{1,2}$:*

- $\mathbb{D}^{15}_{1,2}=\left\{A=\begin{pmatrix}0&0&1&0\\0&0&1&0\end{pmatrix},\ B=\begin{pmatrix}1&0&1&0\\0&1&0&1\end{pmatrix}\right\}.$

3.3 Classification of two-dimensional dendriform algebras over a field of characteristic is three

Theorem 3.4. *Any non-trivial 2-dimensional dendriform algebra over a field $\mathbb{F}$, $(Char(\mathbb{F})=3)$ is isomorphic to only one of the following listed by their matrices of structure constants, such algebras:*

1. *Dendriform algebra generated by $A_{13,3}$:*

- $\mathbb{D}^1_{13,3}(\delta_1)=\left\{A=\begin{pmatrix}0&0&0&0\\1&0&0&0\end{pmatrix},\ B=\begin{pmatrix}0&0&0&0\\\delta_1&0&0&0\end{pmatrix},\ \delta_1\in\mathbb{F},\delta_1\neq 0\right\}.$

2. *Dendriform algebra generated by $A_{7,3}$:*

- $\mathbb{D}^2_{7,3} = \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \right\}.$

3. *Dendriform algebras generated by* $A_{3,3}$*:*

- $\mathbb{D}^3_{3,3} = \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \right\}.$
- $\mathbb{D}^4_{3,3} = \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$
- $\mathbb{D}^5_{3,3}(\delta_4) = \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta_4 \end{pmatrix}, \ \delta_4 \in \{0,1\} \right\}.$
- $\mathbb{D}^6_{3,3} = \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$
- $\mathbb{D}^7_{3,3} = \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$
- $\mathbb{D}^8_{3,3} = \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}.$
- $\mathbb{D}^9_{3,3} = \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \right\}.$
- $\mathbb{D}^{10}_{3,3} = \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix} \right\}.$
- $\mathbb{D}^{11}_{3,3} = \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \right\}.$
- $\mathbb{D}^{12}_{3,3} = \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \right\}.$
- $\mathbb{D}^{13}_{3,3} = \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} \right\}.$

4. *Dendriform algebras generated by* $A_{1,3}$:

- $\mathbb{D}^{14}_{1,3} = \left\{ A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} \right\}.$
- $\mathbb{D}^{15}_{1,3} = \left\{ A = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix} \right\}.$

4 Conclusion

Dendriform algebras offer a rich area of study with connections to various mathematical disciplines. Their unique properties make them valuable for modeling and understanding complex structures in both pure and applied mathematics. As research continues, new applications and deeper insights into these algebras are likely to emerge. Note that those existing classification results on dendriform algebras concern the classification of complex dendriform algebras whereas in the paper we describe the dendriform algebra structures on two-dimensional vector space over any basic field. It is found that on a vector space over a field of the characteristic is not two and three all non-trivial dendriform algebra structures can be assembled into 14 classes whereas the number of such classes in the case of vector spaces over a field of the characteristic is two and three equals 15. It was claimed that in the case of the field of complex numbers the classification result given in [25] must be revised.

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